ECE 346: Digital Signal Processing Spring 2021

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ECE 346: Final Exam Review Guide

Preamble

Thus far, I have made notes for reviewing Exam 1, 2, and the region of convergence. I will not be repeating all of the material from those notes, and will only go over some of the highlights here. As always, note that there may be typos so use it at your own risk. If you spot a mistake, please let me know so that I can edit it.

1 Linear Systems & Signals Review

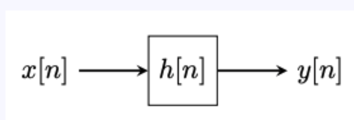
As you would expect, in this section, we will be going over some of the most important properties of linear and time-invariant (LTI) systems. Recall the following:

1. LTI systems can either scale up or scale down the frequencies that are present in an input signal.
2. LTI systems can change the phase of the input frequencies (e.g. $\angle Y(j\Omega) = \angle X(j\Omega) + \angle H(j\Omega)$).
3. Complex exponentials (or sinusoids) are eigenfunctions of LTI systems!

Let's first look at a simple example of using the third property, as many students tend to forget this one.

Example:

Consider the following block diagram of a discrete-time, linear, time-invariant (LTI) system:



Suppose the impulse response $h[n]$ of this LTI system is given by the expression

$$h[n] = \frac{\sin(\frac{\pi}{2}(n-20))}{\pi(n-20)}. \tag{1}$$

Suppose that the input to this system is

$$x[n] = e^{j\frac{\pi}{3}n} - 3\cos\left(\frac{2\pi}{3}n\right). \tag{2}$$

Provide an expression for the output $y[n]$ of this system.

Solution. We want to exploit the third property of LTI systems that **complex exponentials are eigenfunctions of LTI systems**. Thus, the output would look something like

$$y[n] = |H(e^{j\omega_0})| \cdot e^{j(\omega_0 n + \angle H(e^{j\omega_0}))} + |H(e^{j\omega_1})| \cdot e^{j(\omega_1 n + \angle H(e^{j\omega_1}))}, \tag{3}$$

where $H(e^{j\omega})$ is the DTFT of $h[n]$ and ω_0 and ω_1 are the frequencies of $e^{j\frac{\pi}{3}n}$ and $3\cos(\frac{2\pi}{3}n)$, respectively. One should realize here that $H(e^{j\omega})$ is simply a LPF with cutoff frequency $|\omega| \leq \frac{\pi}{2}$ with a phase shift of $e^{-j\omega 20}$. Thus, the cos term of $x[n]$ will be filtered out. For the exponential, we need to evaluate $H(e^{j\omega})$ at $\omega = \frac{\pi}{3}$:

$$H(e^{j\frac{\pi}{3}}) = e^{-j\frac{20}{3}\pi}.$$

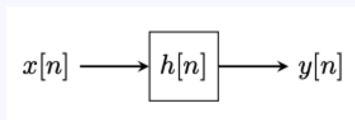
Since $|H(e^{j\frac{\pi}{3}})| = 1$, the final output will have a phase shift of $-\frac{20}{3}\pi$, and hence the output will be

$$y[n] = e^{j(\frac{\pi}{3}n - \frac{20}{3}\pi)}. \tag{4}$$

There are some problems that come in a different form where you can still exploit this property. Let's look at another example:

Example:

Consider the following block diagram of a discrete-time, linear, time-invariant (LTI) system:



Suppose that when the input is $x[n] = 2(-1)^n$, the output is $y[n] = 5(-1)^n$. What would be the output of this filter if the input is $\tilde{x}[n] = 5(-1)^n$?

Solution. Well from what we are given, we know that when

$$x[n] = 2(-1)^n = 2e^{j\pi n}, \tag{5}$$

the output is

$$y[n] = 5(-1)^n = 5e^{j\pi n}. \tag{6}$$

Using the property that complex exponentials are eigenfunctions of LTI systems, that means that

$$|H(e^{j\pi})| = 2.5. \tag{7}$$

Thus, the output of input $\tilde{x}[n] = 5(-1)^n$ would be

$$\begin{aligned}\tilde{y}[n] &= 5 \cdot |H(e^{j\pi})| \cdot e^{j(\pi n + \angle H(e^{j\pi}))} &(8)\\ &= 12.5e^{j\pi n} &(9)\\ &= 12.5(-1)^n. &(10)\end{aligned}$$

2 DTFT & DTFT Properties

The discrete-time Fourier transform (DTFT) should really still be under LSS review, but I decided to make it its own section. In the first exam review guide, I gave a comprehensive list of its properties, so here I highlight the "important" ones. Recall that the DTFT "analysis" and "synthesis" equations are given by

$$X(e^{j\omega}) = \sum_{n=\infty}^{\infty} x[n]\, e^{-j\omega n} \tag{11}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega. \tag{12}$$

The properties I want to mention are

1. **Time Shifting:**

$$x[n] \longleftrightarrow X(e^{j\omega}) \tag{13}$$
$$x[n-n_0] \longleftrightarrow e^{-j\omega n_0} \cdot X(e^{j\omega}) \tag{14}$$

2. **Frequency Shifting (Modulation) Property:**

$$x[n] \longleftrightarrow X(e^{j\omega}) \tag{15}$$
$$e^{j\omega_0 n} \cdot x[n] \longleftrightarrow X(e^{j(\omega-\omega_0)}) \tag{16}$$

3. **Convolution Property:**

$$x[n] \longleftrightarrow X(e^{j\omega}) \tag{17}$$
$$h[n] \longleftrightarrow H(e^{j\omega}) \tag{18}$$
$$x[n] * h[n] \longleftrightarrow X(e^{j\omega}) \cdot H(e^{j\omega}) \tag{19}$$

4. **Parseval's Theorem:**

$$x[n] \longleftrightarrow X(e^{j\omega}) \tag{20}$$
$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 \, d\omega \tag{21}$$

Don't be surprised if you see using any of these properties on your exam!

3 Sampling & Aliasing

The first thing that should come to mind when you hear sampling is the Nyquist criteria:

$$f_s \geq 2 \cdot f_{\max}, \tag{22}$$

where f_s is your sampling frequency and $f_{\max}$ is the bandwidth of your input signal. This also means that your sampling period should satisfy

$$T \leq \frac{1}{2f_{\max}}. \tag{23}$$

There are some problems that ask you for the *constraint* of the sampling period. This is simply asking you to find a period that satisfies this inequality.

In the D/C (discrete-to-continous) block, we also saw how we dealt with the impulse-sampled signal. Recall that if our input signal was $x(t)$ then the CTFT of the impulse sampled signal, $x_p(t)$ is

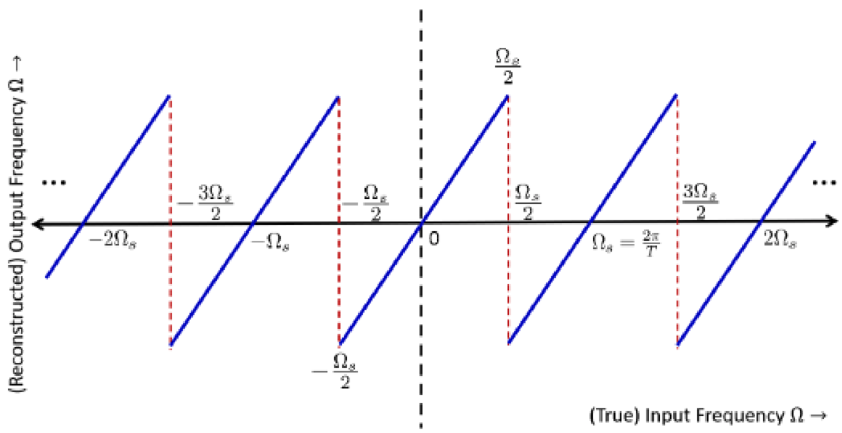
$$x_p(t) = x(t) \cdot p(t) \tag{24}$$
$$X_p(j\Omega) = \frac{1}{2\pi} X(j\Omega) * P(j\Omega) \tag{25}$$
$$X_p(j\Omega) = \frac{1}{2\pi} X(j\Omega) * \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(\Omega - n\Omega_T) \tag{26}$$
$$X_p(j\Omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(j\Omega) * \delta(\Omega - n\Omega_T) \tag{27}$$
$$X_p(j\Omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(j(\Omega - n\Omega_T)). \tag{28}$$

Don't forget the scaling of the $1/T$! If we want to get our continuous-time signal back, we need to put the impulse-sampled signal, say $x_p(t)$ into an *ideal reconstruction filter*. Recall that the ideal reconstruction filter takes the mathematical form

$$H_r(j\Omega) = \begin{cases} T, & |\Omega| \leq \pi/T \\ 0, & \text{otherwise.} \end{cases} \tag{29}$$

The scaling of the T makes sense here, since we scaled by $1/T$ for our impulse-sampled signal.

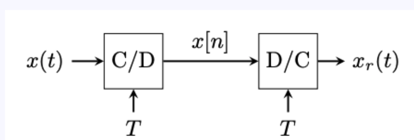
How do we deal with aliasing? There will be some problems that have an anti-aliasing filter that would effectively mitigate aliasing, but sometimes you have to detect which frequencies overlap to which as a result of aliasing. For this, we can use the frequency folding chart:



It seemed like many students had trouble using this chart. Let's do an example:

Example:

Consider the following block diagram of a DSP system:



Suppose the input to this system is given by the following expression:

$$x(t) = 3\cos(100\pi t). \tag{30}$$

If the sampling time was given by $T = \frac{1}{75}$ seconds, determine the output of this system, $x_r(t)$.

Solution. Firstly, note that the Nyquist criteria is not satisfied:

$$200\pi > 150\pi. \tag{31}$$

So where does 100π overlap to? If we identify $\frac{\Omega_s}{2}$ on the frequency folding chart, then we see that $\Omega = 100\pi$ ends up in the negative portion of the chart. Mapping this back onto $[-\frac{\Omega_s}{2}, \frac{\Omega_s}{2}]$, then we see that

$$\pm 100\pi \longrightarrow \mp 50\pi. \tag{32}$$

Hence, our new output is

$$\begin{aligned} x_r(t) &= 3\cos(-50\pi t) & (33) \\ &= 3\cos(50\pi t). & (34) \end{aligned}$$

4 CTFT & DTFT Conversion

We want to define what happens in the frequency domain when going directly from the CTFT to the DTFT (and vice versa):

- When going from $x(t)$ to $x[n]$:

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(j\left(\frac{\omega - 2\pi k}{T}\right)\right), \tag{35}$$

  where $X(e^{j\omega})$ is the DTFT of $x[n]$ and T is the sampling period. The important thing to note here is the scaling of $1/T$.

- When going from $y[n]$ to $y(t)$:

$$Y(j\Omega) = H_r(j\Omega)Y(e^{j\Omega T}), \tag{36}$$

  where $Y(j\Omega)$ is the CTFT of $y(t)$.

- In the case where $H_r(j\Omega)$ above is an **ideal reconstruction filter**:

$$Y(j\Omega) = \begin{cases} TY(e^{j\Omega T}), & |\Omega| \leq \pi/T \\ 0, & \text{otherwise.} \end{cases} \tag{37}$$

Let's do an example:

Example:

Convert the signal

$$X(j\Omega) = 2\pi\delta(\Omega - 100\pi), \tag{38}$$

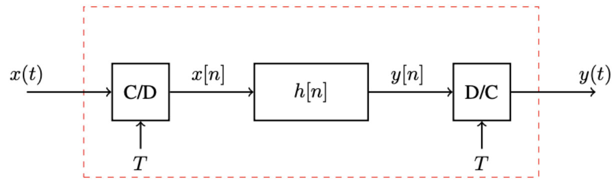
into its DTFT equivalent form using an arbitrary T. You can assume that there is no aliasing.

Solution. We can convert using the following steps:

$$\begin{aligned} X(e^{j\omega}) &= \frac{2\pi}{T}\delta(\Omega - 100\pi)|_{\Omega=\omega/T} & (39) \\ &= \frac{2\pi}{T}\delta\left(\frac{\omega}{T} - 100\pi\right) & (40) \\ &= \frac{2\pi}{T}\delta\left(\frac{1}{T}(\omega - 100\pi T)\right) & (41) \\ &= \frac{2\pi}{T}\cdot T\delta\left(\omega - 100\pi T\right) & (42) \\ &= 2\pi\delta(\omega - 100\pi T). & (43) \end{aligned}$$

For the particular case of delta functions, we do not need to scale! Remember that this is just a special case, and for other functions you do need the scaling of $1/T$.

Now, I want to consider the following block diagram:



Recall that if there is no aliasing occuring in the sampling of $x(t)$, then the red dashed block acts as an LTI system which can be expressed as $H_{\text{eff}}(j\Omega)$:

$$H_{\text{eff}}(j\Omega) = \begin{cases} H(e^{j\Omega T}), & |\Omega| \leq \pi/T \\ 0, & \text{otherwise,} \end{cases} \tag{44}$$

where $H(e^{j\Omega T})$ is the DTFT of $h[n]$ located inside the dashed block. Note here that there is no scaling of T. This makes sense, since the only type of scaling or shifting would be happening due to the $h[n]$ in the red dashed block. Some students ask, "when do we not have the scaling of T?" I guess you can say that this is the case, but don't think of it as formulas you can plug in – you need to reason why there is and isn't scaling.

5 Review of Z-Transforms

The only thing I want to say about z-transforms is the equation

$$X(z) = \sum_{n=\infty}^{\infty} x[n]\, z^{-n}, \tag{45}$$

and that if you know how to determine the region of convergence (ROC), then you should be good to go. Note that if we are given that the DTFT exists, then we can compute the DTFT of the impulse response given the transfer function $H(z)$ as

$$H(e^{j\omega}) = H(z)|_{z=e^{j\omega}}. \tag{46}$$

6 Window-based FIR Filter Design

The general formula for designing window-based filters is the following:

1. State the desired (or ideal) impulse response of the filter, $h_d[n]$. This would be something like one of the filters described above.

2. Shift your filter to make the filter causal:

$$\tilde{h}[n] = h_d[n - M]. \tag{47}$$

3. Window the filter using some windowing method to get the designed filter $h[n]$:

$$h[n] = \tilde{h}[n] \cdot w[n - M], \tag{48}$$

where $w[n]$ is the windowing function.

Many students forgot to also shift the window $w[n]$. I did not include it in my last review guide, and I guess I should have been more explicit. But you should really ask yourself, how does the window make sense if there was no shifting, but $h[n]$ was shifted to be made causal? That is like having a window $w[n]$ that is not causal.

7 Discrete Fourier Transform

Recall that the N-point discrete Fourier transform (DFT) “analysis” and inverse DFT “synthesis” equations are given by

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{kn}, \tag{49}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[K]\, W_N^{-Kn}, \tag{50}$$

where $W_N = e^{-j\frac{2\pi}{N}}$, respectively.

Taking the N-point DFT of a sequence $x[n]$ (i.e. a sum and/or difference of delta functions) is trivial. A quick example:

Example:

Consider the discrete-time signal $x[n]$:

$$x[n] = \delta[n] - 4\delta[n-2]. \tag{51}$$

Compute the 3-point DFT of $x[n]$.

Solution. By explicitly using the DFT formula, note that

$$X[k] = \sum_{n=0}^{2} x[n]\, W_N^{kn} \tag{52}$$

$$= x[0]\, W_N^{k(0)} + x[1]\, W_N^{k(1)} + x[2]\, W_N^{k(2)} \tag{53}$$

$$= 1 \cdot W_N^{k(0)} + 0 \cdot W_N^{k(1)} - 4 \cdot W_N^{k(2)} \tag{54}$$

$$= 1 - 4\, W_N^{2k}. \tag{55}$$

Taking the inverse DFT of a sequence in the same form should be easy as well, as you are just reading off the $x[i]$ terms from $i = 0, \ldots N-1$.

DFT & DTFT

The relationship between the N-point DFT and DTFT is that the DFT is sampling the DFT at points $\omega_k = \frac{2\pi k}{N}$. Note that this is only true for signals that are finite in length (e.g. length of $N-1$). The relationship can be explicitly defined as

$$X[k] = X(e^{j\omega})\big|_{\omega = \frac{2\pi k}{N}}, \tag{56}$$

where $X(e^{j\omega})$ is the DTFT of some finite discrete-time signal $x[n]$. Let’s do a quick example:

Example:

Let the finite discrete-time signal $x[n]$ be defined as

$$x[n] = \delta[n-1] + \delta[n-2] + 3\delta[n-5]. \tag{57}$$

Take the 8-point DFT of this sequence.

Solution. We can easily take the DTFT of this sequence:

$$X(e^{j\omega}) = e^{-j\omega} + e^{-j3\omega} + 3e^{-j5\omega} \tag{58}$$

Evaluating $X[k] = X(e^{j\omega})|_{\omega=\frac{2\pi k}{8}}$,

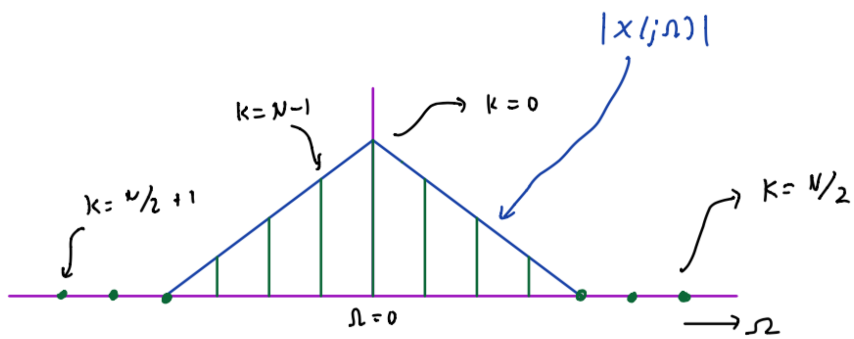
$$X[k] = e^{-j\frac{2\pi k}{8}} + e^{-j\frac{8\pi k}{8}} + 3e^{-j\frac{10\pi k}{8}} \tag{59}$$
$$= W_8^k + W_8^{3k} + 3W_8^{5k}. \tag{60}$$

Spectral Analysis

This is a heavy section that many students get tripped up on. Let's say that we are given the CTFT $X(j\Omega)$ of some signal $x(t)$. We want to find the relationship between $X[k]$ and $X(j\Omega)$. Mathematically, the relationship is the following:

$$X[k] = \begin{cases} \frac{1}{T}X(j\Omega)|_{\Omega=\frac{2\pi k}{NT}}, & k = 0,\dots,N/2 \\ \frac{1}{T}X(j\Omega)|_{\Omega=\frac{2\pi(k-N)}{NT}}, & k = N/2+1,\dots,N-1. \end{cases} \tag{61}$$

Some of you might be used to seeing $\Omega = \frac{2\pi k}{NT} - \frac{2\pi}{T}$, but a little algebra shows that this is also equal to $\Omega = \frac{2\pi(k-N)}{NT}$. I think this picture below sums this up very well:



In the figure above, the blue figure is the magnitude of the CTFT of $x(t)$, and the green bars are taking samples of the CTFT to obtain the DFT. Note that the first half of the DFT corresponds to the positive frequencies of Ω, while the second half correspond to the negative frequencies of Ω. Note that there are some portions of green dots that correspond to $X[k]$=0. How do we determine what position of k's these are? We illustrate this with an example.

Example:

Suppose we had the CTFT $X(j\Omega)$ of a continuous-time signal $x(t)$ that was bandlimited to $\Omega \in [-100\pi, 100\pi]$. Let $N = 40$ and $T = \frac{1}{200}$. Find for which values of k where $X[k] = 0$.

Solution. We should first ask ourselves, for what values of Ω does $X(j\Omega) = 0$? Since $X(j\Omega)$ is bandlimited to 100π, that means for all $\Omega > 100\pi$ and $\Omega < -100\pi$, $X(j\Omega) = 0$. So for the positive frequencies, recall that $X[k] = \frac{1}{T}X(j\Omega)|_{\Omega=\frac{2\pi k}{NT}}$ for $k = 0,\dots,N/2$. Then,

$$\frac{2\pi k}{NT} > 100\pi = \frac{2\pi k \cdot 200}{40} > 100\pi \tag{62}$$
$$= 10\pi k > 100\pi. \tag{63}$$

Solving for k, we get that $k > 10$. That means that for $k = 10,\dots,N/2$, which corresponds to

$k = 10, \ldots, 20$, $X[k] = 0$. We can do the same thing for the negative frequencies:

$$\frac{2\pi(k-N)}{NT} < -100\pi = \frac{2\pi(k-40)\cdot 200}{40} < -100\pi. \tag{64}$$

Solving for k , we get that $k < 30$, and so $X[k] = 0$ for $k = 21, \ldots, 30$.

Lastly, in spectral analysis, we have this notion of **spectral resolution**. The equation for this

$$\frac{1}{NT} \text{ or } \frac{2\pi}{NT}, \tag{65}$$

which is for Hertz and radians, respectively. This resolution is determining how far apart we want our samples of Ω for $X[k]$.

8 Properties of DFT

I want to highlight some of the "important" properties of the DFT. Assuming that we are taking the N -point DFT:

1. **Circular-Time Shifting Property:**

$$x[n] \longleftrightarrow X[k] \tag{66}$$

$$x[\langle n - n_0 \rangle_N] \longleftrightarrow W_N^{kn_0} X[k] \tag{67}$$

2. **Circular Frequency Shifting Property:**

$$x[n] \longleftrightarrow X[k] \tag{68}$$

$$W_N^{-k_0 n} x[n] \longleftrightarrow X[\langle k - k_0 \rangle_N] \tag{69}$$

3. **Circular Convolution Property:**

$$x[n] \textcircled{N} y[n] \longleftrightarrow X[k] \cdot Y[k] \tag{70}$$

4. **Modulation Property:**

$$x_1[n] \cdot x_2[n] \longleftrightarrow \frac{1}{N} X_1[k] \textcircled{N} X_2[k] \tag{71}$$

5. **Parseval's Theorem:**

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=1}^{N-1} |X[k]|^2 \tag{72}$$

Now, we want to ask, when does the circular convolution equal standard convolution (i.e. $x[n] * y[n] = x[n] \textcircled{N} y[n]$)? It turns out that in general, they are not equal, except when

$$N \geq \texttt{length}(x[n] * y[n]), \tag{73}$$

where $\texttt{length}(x[n] * y[n]) = \texttt{length}(x[n]) + \texttt{length}(y[n]) - 1$. If N satisfies this inequality, then the convolutions are equal, and also that

$$\text{N-point DFT of } x[n] \textcircled{N} y[n] = \text{N-point DFT of } x[n] * y[n]. \tag{74}$$

In a similar fashion, we can say that linear convolution can be implemented by taking the following steps:

1. Compute the N-point DFT of $x[n]$ and $y[n]$ with $N \geq \texttt{length}(x[n] * y[n])$
2. Multiply $X[k]$ and $Y[k]$
3. Take the N-point inverse DFT of $X[k] \cdot Y[k]$.

We conclude with some examples.

Example:

Consider the following two sequences:

$$x[n] = -2\delta[n-1] + 4\delta[n-3] \tag{75}$$
$$y[n] = 2\delta[n] + \delta[n-1] + 2\delta[n-2]. \tag{76}$$

Compute $z[n] = x[n] \textcircled{4} y[n]$ using the circular convolution property of the DFT.

Solution. We need to compute $X[k]$ and $Y[k]$ and multiply them:

$$X[k] = -2W_4^k + 4W_4^{3k} \tag{77}$$
$$Y[k] = 2 + W_4^k + 2W_4^{2k} \tag{78}$$

Then,

$$\begin{aligned} Z[k] &= X[k] \cdot Y[k] && (79) \\ &= (-2W_4^k + 4W_4^{3k})(2 + W_4^k + 2W_4^{2k}) && (80) \\ &= 4 + 4W_4^k - 2W_4^{2k} + 4W_4^{3k}. && (81) \end{aligned}$$

Taking the inverse DFT is just reading off of the terms, and thus,

$$z[n] = \{4, 4, -2, 4\}. \tag{82}$$

Example:

Consider the following two sequences:

$$x[n] = \delta[n] = 3\delta[n-2] \tag{83}$$
$$y[n] = \delta[n-1] + \delta[n-3]. \tag{84}$$

Compute the linear convolution, $z[n] = x[n] * y[n]$ by only using the circular convolution property of the DFT.
Solution. Recall that the linear convolution is equivalent to the circular convolution as long as N satisfies

$$N \geq \texttt{length}(x[n] * y[n]). \tag{85}$$

Since the length of $x[n]$ is 3 and the length of $y[n]$ is 4, that means we need $N \geq 6$. Then, following the previous example, we can take the 6-point DFT of $x[n]$ and $y[n]$, multiply them, and then take the inverse DFT. This yields

$$\begin{aligned} Z[k] &= X[k] \cdot Y[k] && (86) \\ &= (1 - 3W_6^{2k})(W_6^k + W_6^{3k}) && (87) \\ &= W_6^k - 2W_6^{3k} - 3W_6^{5k}, && (88) \end{aligned}$$

which yields

$$y[n] = \{0, 1, 0, -2, 0, -3\}. \tag{89}$$

If you wanted to start your sequence from 1, then you should specify that $y[n]$ starts from $n = 1$.